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The Eigenvalue Problem for an N-Sector Ring

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Abstract

The problem of finding the eigenvalues and eigenvectors of a matrix referring to an N-sector ring can be reduced to a number of smaller eigenvalue problems in which the dimensions of the vector space are reduced by a factor N.

I. The Vector Space.

Suppose we need to solve an eigenvalue problem for a matrix A associated with a storage ring, for example in analyzing orbit errors as in Ref.(1). We number the N superperiods with an index ℓ, running over the values 0, ±1, ±2, ... ±(L-1), L, where L = N/2 = 20 for the APS. We assume N is even, as is virtually always the case; the case N odd can be handled without difficulty. It is convenient to let the index ℓ run around the ring in both directions from sector 0 to the diametrically opposite sector L. Sums over ℓ are understood to run over these values. All quantities are taken to be periodic in ℓ, so that ℓ = -L is equivalent to ℓ = L, and ℓ = L+1 is equivalent to ℓ = -(L-1).

We wish to deal with a space of vectors $x_{\ell m}$, where the index m runs over a single superperiod. If, for example, x is the vertical magnet misalignments, then m runs over the magnets in a superperiod. If x is the horizontal orbit displacement from the reference orbit, then m may be a continuous index running from one end of the superperiod to the other, or may be a discrete index running over the places in which we are interested in the orbit. If x is the orbit error, then m may run over the beam position monitors. In the following discussion, we will take m to be an index running over the values 1, 2, ... , M.

A matrix A operating on the vector x will have components denoted by four subscripts: $A_{\ell m \ell' m'}$. We assume that the components $A_{\ell m \ell' m'}$ are real, and that A is symmetric in its two pairs of indices.

II. The Symmetry Operator S.

The ideal ring has N identical superperiods. It is therefore invariant under the operator S which rotates the ring by an angle $2\pi/N$. The operator S has the components

$$S_{\ell m \ell' m'} = \delta_{\ell \ell'-1} \delta_{mm'} . \tag{1}$$

Since S is diagonal in the index m, we will usually omit the index m in this section. It is readily verified that S is orthogonal, i.e.

$$SS^t = 1 , \tag{2}$$

where S^t is the transpose of S.

Since 2L applications of S rotates the ring through 2π, S satisfies the equation

$$(S)^{2L} = 1 . \tag{3}$$

The eigenvalues of S satisfy the same equation and are therefore the $2L^{th}$ roots of 1:

$$s_k = e^{i\pi k/L},\ k = 0, \pm 1, \ldots , \pm(L-1), L . \tag{4}$$

It is convenient to let the eigenvalue label k run over the same sequence of values as ℓ. The eigenfunctions are easily generated from the eigenvalue equation

$$Su^k = s_k \, u^k . \tag{5}$$

If we start by setting the $\ell = 0$ element equal to unity, we get

$$u^k{}_\ell = (s_k)^\ell = e^{i\pi k\ell/L}. \tag{6}$$

These functions can be normalized by dividing each element by $(2L)^{1/2}$.

We note that 40 = 2*2*2*5, so that S^2, S^4, S^5, S^8, S^{10}, S^{20} are also roots of unity, for the APS. We also note that in this case the quantities s_k, $u^k{}_\ell$ are all expressible as linear combinations of the sines and cosines of the four angles $\pi/20$, $\pi/10$, $3\pi/20$, $\pi/5$, and of the angle $\pi/4$, whose sine and cosine are $1/\sqrt{2}$.

III. The Eigenvalue Problem.

We now consider the eigenvalue problem given by the equation

$$Av = av \, . \tag{7}$$

Since A is assumed to describe some property of the ideal ring, it has the 2L-fold symmetry of the ring, and therefore commutes with S:

$$AS = SA \, . \tag{8}$$

This symmetry implies that the components of A can depend on the subscripts ℓ, ℓ' only through the difference $\ell-\ell'$:

$$A_{\ell m \ell' m'} = A_{(\ell'-\ell)mm'}. \tag{9}$$

Since A commutes with S, the eigenvectors v of A can be chosen to be also eigenvectors of S:

$$v = v^{kn} = u^k w^{kn}, \quad n = 1, \ldots , M \, , \tag{10}$$

with elements

$$v^{kn}{}_{\ell m} = u^k{}_\ell \, w^{kn}{}_m = e^{i\pi k\ell/L} \, w^{kn}{}_m. \tag{11}$$

Since A is symmetric and real, its eigenvalues are real. All coefficients in Eq. (7) are then real, and we can find real solutions for the elements of the eigenvector v. The elements (10) are not all real. We conclude that for a complex eigenvalue s of S, there must be two eigenfunctions v and v* of A, corresponding to s and s*, which have the same eigenvalue a. The linear combinations (v+v*)/2 and (v-v*)/2i are then also orthogonal eigenvectors of A corresponding to the degenerate eigenvalue a. These real eigenfunctions are just the real and imaginary parts of the eigenfunctions (10).

We substitute these results into the eigenvalue equation (7) and use Eq.(9):

$$\Sigma_{\ell' m'} A_{(\ell'-\ell)mm'}\, e^{i\pi k\ell'/L}\, w^{kn}{}_{m'} = a_{kn}\, e^{i\pi k\ell/L}\, w^{kn}{}_{m}, \tag{12}$$

$$\Sigma_{rm'} A_{rmm'}\, e^{i\pi kr/L}\, w^{kn}{}_{m'} = a_{kn} w^{kn}{}_{m}. \tag{13}$$

Equation (13) is an eigenvalue problem in the vector space whose components are labelled by m:

$$\Sigma_{m'} B^{k}{}_{mm'}\, w^{kn}{}_{m'} = a_{kn}\, w^{kn}{}_{m}, \tag{14}$$

where

$$B^{k}{}_{mm'} = \Sigma_{r}\, A_{rmm'}\, e^{i\pi kr/L}. \tag{15}$$

Eq. (14) is an eigenvalue problem in a complex M-dimensional space labelled by the index m. Since $B^{k}{}_{mm'}$ is hermitian, it has M real eigenvalues a_{kn}. Note that $B^{-k}{}_{mm'} = (B^{k}{}_{mm'})^{*}$, so $B^{-k}{}_{mm'}$ has the same eigenvalues a_{kn}, and $w^{-kn}{}_{m} = w^{-kn}{}_{m}{}^{*}$.

REFERENCES

1. Keith Symon, "The Determination of Orbit Errors and Corrections in Particle Accelerators", UW-SRC-63.